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SCATTERING OF ACOUSTICAL WAVES
BY A SPINNING ATMOSPHERIC TURBULE

September 1992

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Under Contract DAAD07-86-C-0034
Contract Monitor Harry J. Auvermann

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Theory is developed for acoustical scattering by localized quasistatic atmospheric turbulent structures (turbules) that contain both flow and density variations. The quasi-static density variation is shown to be proportional to v^2, where v is the (presumably small) ratio of flow speed to asymptotic acoustical wavespeed. The flow contributes terms $O(v^2)$ to the scattering cross sections, while the density varia tions contribute terms $O(v^4)$. Differential and total cross sections are calculated for a Gaussian spinning turbule model of effective radius $\underline{a}$, for size parameters ka from 0.25 to 2.0, where $k = 2\pi/\lambda$, λ = wavelength, using the first Born approximation, which is estimated to be valid for ka 2 for $v_{max} < 0.1$. The contributions of the density variation are about three orders of magnitude smaller than those of the flow. The latter are proportional to $(ka)^6$ and $\sin^2\theta\cos 2\ \theta\sin^2\theta_\alpha\sin(\phi-\phi_\alpha)$, where (θ,ϕ) are theP (polar, azimuthal) scattering angles, and $(\theta_\alpha,\phi_\alpha)$ are the spin axis angles. The contributions of the density variations go like $(ka)^4$ and have a somewhat Rayleigh-like dependence on θ. A digitized Green function approach and computer algorithm are also developed, for the $O(v^2)$ flow contributions only. In principle, this method would allow treatment of turbules of arbitrary structure, shape, and size.

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I. INTRODUCTION

Description of the scattering of both acoustic and electromagnetic waves by localized turbulent structures (turbules) in the atmosphere is currently of great interest. The usual model of an individual turbule consists of a localized region in which the atmospheric mass density differs from that of the surroundings; such a model can be described in terms of a variable refractive index. (See, for example, Ishimaru 1978). The principal purpose of the work reported here was to investigate the effect on acoustic scattering of including realistic flow velocities in a model turbule; to the best of the author's knowledge, this has not been done previously. In order to achieve this purpose, it was first necessary to develop the appropriate acoustical wave equations and descriptions of the scattering from the fundamental fluid equations.

This report is organized as follows. In section II, the above-mentioned theoretical structure that was developed is presented, along with the appropriate scattering theory. The basic fluid equations were chosen to be those for an ideal fluid with negligible heat flow. The principal results quoted in this section are i) that the logarithm of the ratio of the fluid particle number density inside a quasistatic turbule to that of the surroundings is proportional to the square of the ratio of the turbule flow speed to the acoustic wavespeed in the surroundings, in the absence of sound waves, and ii) the acoustical wave equations, in terms of small space and time-dependent variations in number density and fluid velocity with respect to their local quasistatic values, include terms that are first order in this ratio. Since this ratio is generally quite small, the presence of such first-order terms in the wave equations presages the possibility that the contributions of these terms to the scattering might dominate those of the density variations.

In section III, the application of the theory to a realistic quasistatic nonuniformly spinning turbule is presented. Two approaches to calculating the acoustical scattering are discussed. One is the Born approximation. Analysis is given to show that this approximation should be reasonably valid for turbule size parameters $ka < 5$ and very good for size parameters < 1. (Here, $k=2\pi/$(incident wavelength), a = effective radius). Another is the digitized Green function (DGF) method, which should be applicable to a turbule of arbitrary structure and size (Goedecke and O'Brien, 1988). However, during the course of the work reported, the author was unable to apply this method to the complete equations; its application is thus far restricted to the equations in which only the first-order terms in the above-mentioned ratio are retained. The principal results quoted in this section are the Born approximation scattering efficiency vs. scattering angle for several values of size parameter, and of total scattering efficiency vs.

size parameter. The major findings are that i) the above-mentioned first-order terms in the wave equations produce scattering cross sections about three orders of magnitude larger than do the terms involving density changes alone, for realistic values of the flow parameters, over a range of size parameters from 0.25 to 2.0, and ii) these first-order efficiencies are proportional to $(ka)^6$, while the efficiencies due to density variability have the expected $(ka)^4$ dependence, for $ka < 2$.

In section IV, a brief discussion is given of the results of the work performed and of needs for future development. In particular, it is suggested that even more realistic quasistatic turbule models should be investigated. These include "smoke rings," i.e., toroidal flow patterns. But this requires another development from first principles, because such vortices must translate as a whole at constant speed in still air in order to exist. Also, it is suggested that the DGF method be developed further and eikonal methods also be developed for application to the problem.

II. THEORY

A. Fluid Equations

1. General. We consider a fluid with negligible viscosity and heat conduction. For simplicity, we let the fluid be a single-component ideal gas; extension to a multi-component fluid is straightforward. The fluid equations involve the number density $n(\vec{r},t)$, the velocity field $(\vec{V}(\vec{r},t)$, the pressure $p(\vec{r},t)$, the temperature $T(\vec{r},t)$, the internal energy density $U(\vec{r},t)$, and the mass m of the atoms, as follows:

$$\partial n/\partial t + \vec{\nabla}\cdot(n\vec{V}) = 0 \tag{1}$$

$$\partial \vec{V}/\partial t + \vec{V}\cdot\vec{\nabla}\vec{V} = -\vec{\nabla}p/mn \tag{2}$$

$$\partial U/\partial t + \nabla\cdot(\vec{V}U) + p\ \vec{\nabla}\cdot\vec{V} = 0 \tag{3}$$

$$p = nk_BT = \Gamma U, \quad \Gamma \equiv \gamma-1, \tag{4}$$

$$\gamma = c_p/c_v = \text{ratio of specific heats}$$

The first two equations are those of number (mass) continuity and the Euler equation for an ideal fluid. The third is the thermal energy flow equation; it results from the more complete energy continuity relation

$$\partial_t(\tfrac{1}{2}nmV^2 + U) + \vec{\nabla}\cdot[\vec{V}(U+p) + \tfrac{1}{2}mnV^2\vec{V} + \vec{Q}] = 0, \tag{5}$$

when the heat flux $\vec{Q} \equiv 0$, in view of eqs. (1) and (2). Eq. (4) is the ideal gas equation of state and the expression of equipartition for the model fluid.

Combining (1), (3), (4) yields

$$T^{-1}\ (\partial T/\partial t + \vec{V}\cdot\vec{\nabla}T) + \Gamma\vec{\nabla}\cdot\vec{V} = 0 \tag{6}$$

Rewriting (1) yields

$$\Gamma n^{-1}\ (\partial n/\partial t + \vec{V}\cdot\vec{\nabla}n) + \Gamma\vec{\nabla}\cdot\vec{V} = 0 \tag{7}$$

Combining (6) and (7) yields

$$\partial_t\ [\Gamma\ \ell n(n) - \ell nT] + \vec{V}\cdot\vec{\nabla}[\Gamma\ell n(n) - \ell nT] = 0 \tag{8}$$

Since we have assumed zero heat flux (adiabatic conditions), we should have

$$p = Cn^{\gamma}, \quad c = \text{constant} \tag{9}$$

But, since $p = nk_BT$ also, we have

$$T = Cn^{\Gamma}/k_B \tag{10}$$

But then eq. (8) is satisfied identically. Thus, we can eliminate T from eqs. (1)-(4). In particular, we may write (2) as

$$\partial\vec{V}/\partial t + \vec{V}\cdot\vec{\nabla}\vec{V} = -(\gamma k_BT/m)n^{-1}\vec{\nabla}n = -(\gamma C/\Gamma m)\vec{\nabla}n^{\Gamma} \tag{11}$$

We assume that the fluid contains a localized region R centered at the origin of coordinates in which there is a quasistatic departure of n and $\vec{V}$ from their uniform constant background values at spatial infinity. Also, we assume that there can be a small-amplitude wavelike disturbance in n and $\vec{V}$ everywhere in the fluid. That is, we assume

$$n(r,t) = n_o(\vec{r})(1 + \epsilon(\vec{r},t)), \tag{12}$$

$$\vec{V}(\vec{r},t) = \vec{v}(\vec{r}) + \vec{u}(\vec{r},t), \tag{13}$$

where $|\epsilon| \ll 1$ and $|\vec{u}|$ is "small." Also, we assume that as $r\to\infty$,

$$n_o(r) \to n_\infty = \text{const}, \quad \vec{v}(r) \to 0. \tag{14}$$

The form of eqs. (7) and (11) suggests that we define

$$\ell n(n/n_\infty) \approx \psi(\vec{r}) + \epsilon(\vec{r},t), \tag{15}$$

where we've used (12) and kept terms to first order in $\epsilon(\vec{r},t)$ only, and $\psi(\vec{r})$ is defined by

$$\psi(\vec{r}) = \ell n(n_o(\vec{r})/n_\infty) \tag{16}$$

As we shall see below, $|\psi(\vec{r})| \ll 1$ for realistic turbules. So, when we substitute (12, 13, 15) into the fluid equations (1) and (11), keep terms of order $(\epsilon, \vec{u}, \psi, v^2, \epsilon\psi, \vec{u}\psi)$ only, and require that the time-dependent terms and the static terms in the equations be separately equated, we get

$$\vec{v}\cdot\vec{\nabla}\vec{v} = -c^2_\infty\vec{\nabla}\psi \tag{17}$$

$$\vec{v}\cdot\vec{\nabla}\psi + \vec{\nabla}\cdot\vec{v} = 0$$

$$\partial\vec{u}/\partial t + \vec{v}\cdot\vec{\nabla}\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{v} = -c^2_\infty\vec{\nabla}(\Lambda\epsilon) \tag{18}$$

$$\partial\epsilon/\partial t + \vec{v}\cdot\vec{\nabla}\epsilon + \vec{u}\cdot\vec{\nabla}\psi + \vec{\nabla}\cdot\vec{u} = 0,$$

where

$$\Lambda = 1 + \Gamma\psi \tag{19}$$

and

$$c^2_\infty = \gamma k_B T_\infty / m \tag{20}$$

is the square of the asymptotic wavespeed. Equations (17) are the static (zeroth order in ϵ,u) equations that a quasistatic turbule's motion and density should satisfy, for adiabatic conditions. Equations (18) are the acoustic equations for a wavelike disturbance in the presence of the turbule.

Now, as $r \to \infty$, $\psi=0$, $\vec{v}=0$, so, if $\vec{\nabla} x \vec{v} = 0$, we have from

$$\psi = - v^2/2c^2_\infty \tag{21}$$

For cases of interest, $v/c_\infty << 1$, so $|\psi| <<< 1$; ψ is <u>second order</u> in the small ratio v/c_∞. If $\vec{\nabla} x \vec{v} \neq 0$, which is likely, it is still clear from (17) that $|\psi|$ is of order v^2/c^2_∞, for if $\vec{v} \equiv 0$ everywhere, then, in view of the boundary conditions, $\psi \equiv 0$ everywhere.

In the acoustical equations (18), we may consider time dependence $e^{-i\omega t}$, since these equations are linear in $(\epsilon, \vec{u})$ and none of the coefficients depend on time. Thus, we get

$$-i\omega\vec{u} + C^2_\infty \vec{\nabla}(\Lambda\epsilon) + \vec{v}\cdot\vec{\nabla}\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{v} = 0 \tag{22}$$

$$-i\omega\epsilon + \vec{\nabla}\cdot\vec{u} + \vec{v}\cdot\vec{\nabla}\epsilon + \vec{u}\cdot\vec{\nabla}\psi = 0$$

If we are given $\psi(\vec{r})$ and $\vec{v}(\vec{r})$, localized in R, and given a plane wave incident from infinity, then in principle we can solve these equations for $(\epsilon, \vec{u})$ inside R, and then predict how the incident wave scatters.

2. <u>Wave equations</u>. The first step in obtaining wave equations of the appropriate form for describing scattering is to decouple eqs. (22). This is algebraically difficult, but becomes reasonably simple if we consistently discard terms of order v^3 and higher, remembering that ψ is of order v^2. In the ensuing manipulation, it helps immensely if we put the equations in dimensionless form. To do this, we define or redefine several quantities, as follows:

$$k \equiv \omega/c_\infty = 2\pi/\lambda, \ \lambda = \text{incident wavelength}$$

$$r \to kr \equiv r; \ \vec{\nabla} \to k\vec{\nabla}_{(kr)} \equiv k\vec{\nabla} \tag{23}$$

$$\vec{v}/c_\infty \to \vec{v}; \ \vec{u}/c_\infty \to \vec{u}$$

$$\delta = \Lambda\epsilon$$

With these substitutions, and keeping only terms up to $O(v^2)$, eqs. (22) become

$$\vec{u} = -i\vec{\nabla}\delta + \vec{F}$$
$$\delta = -i[\Lambda\vec{\nabla}\cdot\vec{u} + \vec{v}\cdot\vec{\nabla}\delta + \vec{u}\cdot\vec{\nabla}\psi] \tag{24}$$

where

$$\vec{F} \equiv -\ i(\vec{v}\cdot\vec{\nabla}\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{v}) \tag{24a}$$

It is easiest to eliminate δ from eqs. (24). After some algebra, the following equation results:

$$(\nabla^2+1)\vec{u} = (\nabla^2+1)\vec{F} + \vec{\nabla}(U - \vec{\nabla}\cdot\vec{F}) \equiv \vec{S}(r) \tag{25}$$

where

$$U \equiv -\ \vec{u}\cdot\vec{\nabla}\psi - \Gamma\psi\vec{\nabla}\cdot\vec{u} - i\vec{u}\cdot\vec{v} + i\vec{v}\cdot\vec{F} \tag{26}$$

Eq. (25) is in the "standard form" that allows a Green's function treatment of the scattering problem; the source function $\vec{S}$ goes to zero rapidly for $r>a$.

B. Scattering Theory

1. General. We consider a plane wave of unit amplitude for the relative density disturbance $\epsilon(\vec{r})$ incident from infinity in direction $\hat{k}$. Then

$$\epsilon_{in}(r) = \exp(i\hat{k}\cdot\vec{r}) \rightarrow u_{in}(\vec{r}) = \hat{k}\exp(i\hat{k}\cdot\vec{r}) \tag{27}$$

This form for $u_{in}(\vec{r})$ follows from eqs. (22) with $\psi \equiv 0$, $\vec{v} = 0$. Now, we write down the Green's function implicit solution for eq. (25), for outgoing scattered waves:

$$u(\vec{r}) = u^{in}(\vec{r}) - (4\pi)^{-1}\int d^3r'(e^{iR}/R)S(\vec{r}\,') \tag{28}$$

where

$$R \equiv |\vec{r} - \vec{r}\,'| \tag{29}$$

and the integral on $\vec{r}\,'$ is over all space.

The second term on the RHS of (28) is the scattered wave $\vec{u}_s(\vec{r})$. For $r \rightarrow \infty$, we get

$$\vec{u}_s(r) \xrightarrow[r\to\infty]{} (ke^{ir}/r)\hat{r}f(\vec{k},\hat{r}) \tag{30}$$

where the scattering amplitude $f(\vec{k},\hat{r})$ is

$$f(\vec{k},\hat{r}) = -(4\pi k)^{-1}\hat{r}\cdot\int d^3r' \exp(-i\hat{r}\cdot\vec{r}')\vec{S}(r') \tag{31}$$

The form (30) of the scattered wave for $r \to \infty$ follows because $\vec{u}_s(\vec{r})$ must also satisfy eqs. (21) for $\psi=0$, $\vec{v}=0$; so we must have for $r\to\infty$

$$u_s(r) = i\vec{\nabla}\epsilon_s(\vec{r}) = \hat{r}\epsilon_s(\vec{r}). \tag{32}$$

This says that the outgoing spherical scattered wave must be longitudinal.

It is straightforward to show that the standard definition of the differential scattering cross section, equal to the average (acoustical) power scattered per unit solid angle divided by the incident power per unit area, leads to the identification of the relevant cross sections just as in standard scattering theory in quantum mechanics or electromagnetism.

In the scattering of electromagnetic waves by a spherical dielectric particle of radius a, the important quantities are the size parameter, $ka = 2\pi a/\lambda$, and the scattering efficiencies, which are equal to the scattering cross sections divided by the geometrical cross section πa^2. In our treatment of the acoustical problem, an effective radius a will be a parameter of any model turbule. Thus, the important scattering efficiencies will be the following:

$$Q(\vec{k},\hat{r}) = |f(\vec{k},\hat{r})|^2/\pi a^2 = \text{differential scattering efficiency} \tag{33}$$

$$Q_s(\vec{k}) = \int d\Omega Q(\vec{k},\hat{r}) = \text{total scattering efficiency} \tag{34}$$

$$Q_e(\vec{k}) = (4/ka^2)\ \mathrm{Im}(f(\vec{k},\hat{k})) = \text{extinction efficiency} \tag{35}$$

Eq. (35) is part of the "optical theorem," which states that $Q_e = Q_s + Q_a$, Q_a = absorption efficiency. Here, $Q_e = Q_s$, because we have not included viscosity, so there is no absorption.

If we substitute eq. (25) for S into eq. (31), we get immediately on integration by parts

$$f(\vec{k},\hat{r}) = -i(4\pi k)^{-1} \int d^3r'(U(\vec{r}')-\vec{\nabla}'\cdot\vec{F}(\vec{r}'))\exp(-i\hat{r}\cdot\vec{r}'). \tag{36}$$

Unfortunately, we cannot simply evaluate this integral, because the integrand contains the unknown wave $\vec{u}(\vec{r}')$ and its derivatives. In what follows, we first derive the Born approximation to this problem; then we derive a digitized Green's function approach (Goedecke and O'Brien, 1988).

2. First Born Approximation. This consists of replacing the actual wave $\vec{u}(\vec{r}')$ by the incident wave $\vec{u}_{in}(\vec{r}')$ wherever the former occurs in the integrand of (36). In the corresponding problem in electromagnetic scattering by dielectric particles of average

radius $\underline{a}$, refractive index m, this approximation is valid only for $|m-1|ka \ll 1$. We don't have a true "refractive index" in this problem; but in what follows we will specify conditions under which this approximation should be valid for this acoustic scattering problem.

We have treated only cases for which $\vec{\nabla}\cdot\vec{v} = 0$, so what follows is valid only for this condition. After considerable algebra, we get from (24a, 26, 36)

$$f = f^{(1)} + f^{(2)}{}_n + f^{(2)}{}_v \tag{37}$$

where

$$f^{(1)} = -\ (2\pi k)^{-1} \cos\theta \hat{r}_i I^{(1)}{}_i$$

$$f^{(2)}{}_n = +(4\pi k)^{-1}(1-\gamma + 1-\cos\theta) I^{(2)}{}_n \tag{38}$$

$$f^{(2)}{}_v = -\ (4\pi k)^{-1}\{(1-\cos\theta)(I^{(2)}{}_v)_{ii} - \hat{k}_i\hat{k}_j(I^{(2)}{}_v)_{ij}\}$$

where summation convention is used, and

$$I^{(1)}{}_i \equiv \int d^3r' v_i(\vec{r}')\ \exp(i\vec{K}\cdot\vec{r}')$$

$$I^{(2)}{}_n \equiv \int d^3r' \psi(\vec{r}') \exp(i\vec{K}\cdot\vec{r}') \tag{39}$$

$$(I^{(2)}{}_v)_{ij} \equiv \tfrac{1}{2}\int d^3r' v_i(r') v_j(r') \exp(i\vec{K}\cdot\vec{r}')$$

where

$$\vec{K} \equiv \hat{k}-\hat{r} \tag{40}$$

Here, $f^{(1)}$ is the contribution that is first order in the turbule velocity $\vec{v}$; $f^{(2)}{}_v$ is second order in this; and $f^{(2)}{}_n$ is the only contribution that would occur if the turbule density were nonuniform but the flow $\vec{v}$ were set equal to zero.

This is as far as we can go without specifying a flow $\vec{v}(\vec{r})$ and a logarithmic density variation $\psi(\vec{r})$ for the turbule. We report this in the next section.

3. Digitized Green Function Solution. This method was originally developed to allow modelling of electromagnetic scattering by dielectric particles of arbitrary shape and morphology (Goedecke and O'Brien 19C8). The method would normally begin from the causal Green's function solution (28) of (25). In order to use the DGF

method, we would have to reduce the Green's function implicit solution first to the form

$$u_i(\vec{r}) = u_{in,i}(\vec{r}) + \int d^3r' K_{ij}(\vec{r},\vec{r}')u_j(\vec{r}') \tag{41}$$

where K_{ij} is a known function involving $\vec{v}$, ψ, and the Green's function, and their derivatives. We have not yet been able to do this.

We have been able to express a Green's function solution for $\epsilon(\vec{r})$, valid only to terms of first order in $\vec{v}$, in this form. The result for $\vec{\nabla}\cdot\vec{v} = 0$ is

$$\epsilon(\vec{r}) = \epsilon_{in}(\vec{r}) + \int d^3r' K(\vec{r},\vec{r}')\epsilon(\vec{r}') \tag{42}$$

where

$$K = K_A + K_B \tag{43}$$

$$K_A(\vec{r},\vec{r}') = (4\pi)^{-1} e^{iR}\hat{R}_i(iR^{-1}-R^{-2})(2iv_i(\vec{r}') + \partial'_j\partial'_j v_i(\vec{r})) \tag{44}$$

$$K_B(\vec{r},\vec{r}') = (4\pi)^{-1} e^{iR}(-iR^{-1} -3iR^{-2}+3R^{-3})\hat{R}_i\hat{R}_j(\partial_j' v_i(\vec{r}')) \tag{45}$$

This result is usable, since we know that v<<1 in application, so the first order terms in $\vec{v}$ should usually dominate second order contributions to the scattering.

In order to proceed with the DGF method, we have to choose analytic expressions for $v_i(\vec{r}')$, so that we may evaluate the derivatives $\partial'_j v_i(\vec{r}')$, $\partial_j'\partial_j' v_i(\vec{r}')$ at any point r'. Then $K(\vec{r},\vec{r}')$ is a known function. Then we subdivide the region R into cubical cells of some side length $\underline{d}$, with cell number $\underline{\alpha}$ centered at $\vec{r} = \vec{r}_\alpha$. Then eq. (45) becomes a matrix equation,

$$\epsilon_\alpha = \epsilon_{in,\alpha} + K_{\alpha\beta}\ \epsilon_\beta \tag{46}$$

in obvious summation notation, where

$$K_{\alpha\beta} = d^3K(\vec{r}_\alpha,\vec{r}_\beta). \tag{47}$$

Then we invert this matrix equation numerically, given $\epsilon_{in,\alpha} = \exp(i\vec{k}\cdot\vec{r}_\alpha)$, and obtain values for the ϵ_α, taking care with the self-terms $K_{\alpha\alpha}$. Once the ϵ_α are known, we can evaluate the scattering amplitude easily, and then obtain the cross sections equally easily. For this formulation, a little algebra yields for the scattering amplitude

$$f(\vec{k},\hat{r}) = f_A + f_B \tag{48}$$

where

$$f_A(\vec{k},\hat{r}) = id^3(4\pi k)^{-1}\hat{r}_i\ \Sigma_\alpha \exp\ (-i\hat{r}\cdot\vec{r}_\alpha)(2iv_i(\vec{r}_\alpha)\ +\nabla^2 v_i(\vec{r}_\alpha))\epsilon_\alpha$$

$$f_B(\vec{k},\hat{r}) = id^3(4\pi k)^{-1}\hat{r}_i\hat{r}_j\ \Sigma_\alpha \exp(-i\hat{r}\cdot\vec{r}_\alpha)(\partial_j v_i(\vec{r}_\alpha))\epsilon_\alpha$$

Experience with the DGF method in electromagnetic scattering has shown that, in order to achieve accurate results, the cell side length d must be chosen $\leq |m|^{-1}$, where m ≡ refractive index of scatterer. (Remember, here d is in units of (incident wavelength/2π)). If the average linear extension of a turbule is a (length units), then the number of cells needed is $\geq(|m|ka)^3$. It's not clear here just what to use for an effective refractive index; but, roughly speaking, the total wavespeed at any point r in the turbule lies between $c_\infty(1 + v(r))$ and $c_\infty(1-v(r))$. Thus, the effective refractive index satisfies $1-v(r) \leq m_{eff} \leq 1 - v(r)$. Since $v(r) << 1$ in applications here, then, the number of cells needed for accuracy with the DGF method must be $\geq (ka)^3$. Clearly, for large size parameters ka, the inversion of (46) can be very demanding of computer storage and time.

III. APPLICATIONS AND RESULTS

A. Model Turbule.

We have considered a nonuniformly spinning turbule model. We chose the following function for the turbule flow,

$$\vec{v}(\vec{r}) = (\vec{\alpha} X \vec{r}) \exp(-r^2/(ka)^2) \tag{49}$$

where

$$\vec{\alpha} \equiv \omega^{-1}\vec{\Omega} = \hat{\alpha} v_a (ka)^{-1}, \quad v_a \equiv \Omega a/c_\infty, \tag{50}$$

so that Ω is an angular velocity parameter.

The Gaussian is similar to what has been used recently for density variations in a turbule with v=0 (McBride, 1989). The length parameter a is a measure of the "radius" of the turbule.

Eqs. (17) should be satisfied. However, we have been unable to find any spatially limited solutions of the first of these equations for (ψ, v). With the above v, we used the result that would obtain if $\nabla x v$ were equal to zero,

$$\psi = -v^2/2 = -\tfrac{1}{2}\, \alpha^2 (r^2 - (\hat{\alpha}\cdot\vec{r})^2) \exp(-2r^2/(ka)^2), \tag{51}$$

as a simple choice for ψ which satisfies continuity and has the correct magnitude, but does not satisfy the first of eqs. (17). The results should be indicative of the magnitude and angular dependence of the scattering cross sections of actual turbules, as long as this model is quasistatic, i.e., as long as this density variation and velocity structure persist for a time that is long compared to the period of the incident wave. We assume that this is the case. We note that the above-mentioned density variations alone that have been used for a (quasistatic) turbule cannot persist either, according to eq. (17), for if $\vec{v}=0$, then $\vec{\nabla}\psi=0$.

B. Born Approximation

1. Expressions for Scattering Amplitude. We substitute the velocity $\vec{v}$ and disturbance ψ of eqs. (49 and 51) into Eqs. (39). After some algebra, the contributions to the scattering amplitude of eqs. (38) are found to be

$$f^{(1)}(\vec{k},\hat{r}) = -(\alpha/2\pi k)\cos\theta\, \epsilon_{jki}\hat{r}_j\hat{\alpha}_k J_i \tag{52}$$

$$f^{(2)}{}_n(\vec{k},\hat{r}) = (\alpha^2/4\pi k)(1-\gamma + K^2/2)\tfrac{1}{2}(\delta_{ij} - \hat{\alpha}_i\hat{\alpha}_j) J_{ij} \tag{53}$$

$$f^{(2)}{}_v(k,\hat{r}) = (\alpha^2/4\pi k)[(K^2/4)(\delta_{ij}-\hat{\alpha}_i\hat{\alpha}_j) + \hat{k}_r\hat{k}_s\hat{\alpha}_t\hat{\alpha}_p\epsilon_{rti}\epsilon_{spj}]J_{ij} \tag{54}$$

where

$$J_i \equiv i\tfrac{1}{2}\pi^{3/2}(ka)^5\ K_i\ \exp(-(ka)^2K^2/4) \tag{55}$$

$$J_{ij} \equiv \tfrac{1}{4}(ka)^5\ [\delta_{ij} - \tfrac{1}{4}(ka)^2\ K_iK_j](\pi/2)^{3/2}\ \exp(-(ka)^2K^2/8) \tag{56}$$

and, as in eq. (40),

$$K_i \equiv \hat{k}_i - \hat{r}_i \quad , \ K^2 = 2(1-\cos\theta) \tag{57}$$

where θ is the polar scattering angle, such that $\hat{k}\cdot\hat{r} = \cos\theta$. Note that the observation direction is specified by the unit radial vector $\hat{r}$,

$$\hat{r} = \vec{e}_1\ \sin\theta\cos\phi + \vec{e}_2\ \sin\theta\cos\phi + \vec{e}_3\cos\theta \tag{58}$$

in terms of the angle θ and the azimuthal angle ϕ, where the $\vec{e}_i$ are the Cartesian unit basis vectors.

With no loss of generality, we may take the unit incident wave vector to be in the z-direction,

$$\hat{k} = \vec{e}_3 \tag{59}$$

The orientation of the spin axis of the model turbule is then arbitrary, so we have the unit spin axis vector given by

$$\hat{\alpha} = \vec{e}_1\ \sin\theta_\alpha\cos\phi_\alpha + \vec{e}_2\sin\theta_\alpha\sin\phi_\alpha + \vec{e}_3\ \cos\theta_\alpha \tag{60}$$

In practice, it is easy to see that all differential cross sections will depend on the difference $(\phi-\phi_\alpha)$ only. Thus, we can put $\phi_\alpha = 0$ during calculation, and then simply replace ϕ by $\phi-\phi_\alpha$ in the results.

As discussed earlier, the (first) Born approximation should be valid if the effective refractive index m satisfies $|m-1|ka \ll 1$. Here, the "effective index" can probably be taken as $m \approx 1 \pm v_a$, as discussed in the previous section on the DGF method. Thus, the criterion for validity is

$$v_a ka \ll 1 \tag{61}$$

In practice, $|m-1|ka$ up to unity is often used. For realistic models, $v_a < 0.2$, so the Born approximation results should be fairly trustworthy for $ka < 5$, and very good for $ka < 1$, and better the smaller v_a.

2. First Order Efficiencies. If we combine eqs. (33, 34, 50, 52, 55, and 57-60), we obtain

$$Q^{(1)}(\vec{k}\hat{r}) = (v^2_a/16)(ka)^6\sin^2\theta\cos^2\theta\sin^2\theta_\alpha\sin^2(\phi-\phi_\alpha) \tag{62}$$

$$X \exp[-(ka)^2(1-\cos\theta)]$$

This has rather striking dependence on the observation direction and on the turbule spin axis orientation. It goes to zero if i) $\theta_\alpha = 0$ or π, ii) $\phi = \phi_\alpha$ or $\phi_\alpha \pm \pi$, iii) $\theta = 0, \pi/2, \pi$. For example, suppose that $\theta_\alpha = \pi/2$, $\phi_\alpha = 0$. Then the scattering is zero at at $\phi = (0,\pi)$ and $\theta = (0,\pi/2, \pi)$, and is max at $\phi = (\pi/2, 3\pi/2)$ and $\theta = (\pi/4, 3\pi/4)$. For $ka > 1$, the exponential becomes important, and strongly reduces scattering in the backward hemisphere relative to the forward.

This differential efficiency also has an unusual dependence on ka, going as $(ka)^6$, quite different from Rayleigh scattering, which goes like $(ka)^4$ with very different dependence on scattering angle.

It is of great interest to consider the efficiency averaged over random orientations of the spin axis. Since

$$(4\pi)^{-1}\int d\Omega_\alpha \sin^2\theta_\alpha \sin^2(\phi-\phi_\alpha) = 1/3 \tag{63}$$

We get

$$\langle Q^{(1)}(\vec{k},\hat{r})\rangle = (v^2_a/48)(ka)^6 \sin^2\theta\cos^2\theta \exp[-(ka)^2(1-\cos\theta)] \tag{64}$$

This is plotted vs. θ for several different values of ka in Figs. 1 and 2, for $v_a = 0.1$.

The total scattering efficiencies are obtained by integrating (62 and 64) over all observation angles. This can be done analytically. After some algebra, we get

$$\langle Q^{(1)}{}_s(\vec{k})\rangle = (\pi v^2_a/6)(ka)^6 \quad \Sigma_{n=1}^{4} b_n I_n((ka)^2) \tag{65}$$

where

$$b_1 = 0.5,\ b_2 = -1.25,\ b_3 = 1,\ b_4 = -0.25 \tag{66}$$

and

$$I_n(x) \equiv \int_0^2 d\xi\ \xi^n\ e^{-x\xi} \tag{67}$$

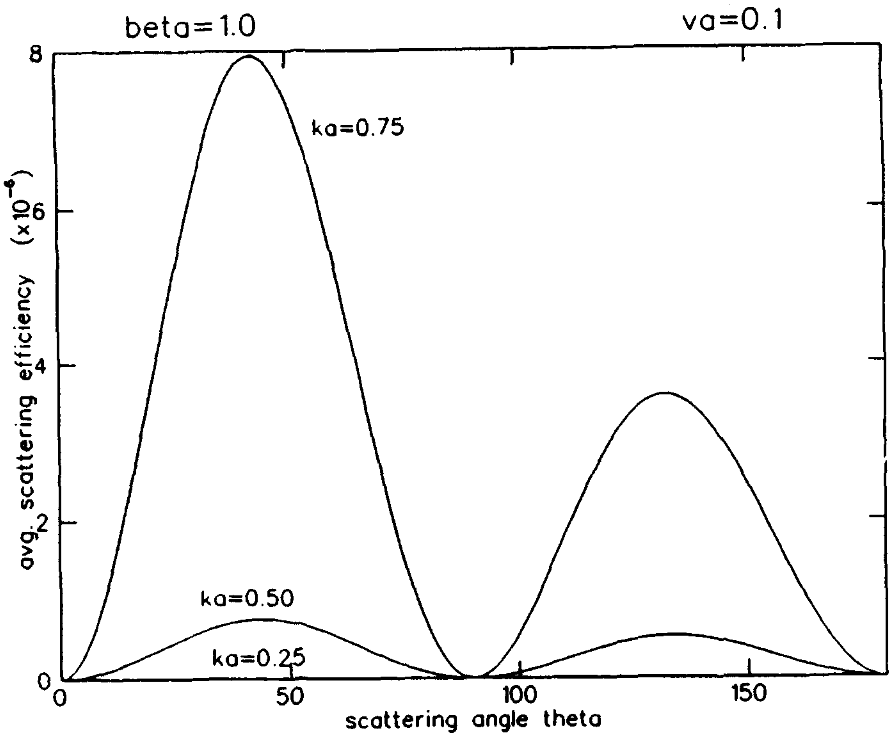

Figure 1. Scattering efficiency vs. scattering angle for a spinning turbule, including only first order contributions for small size parameters.

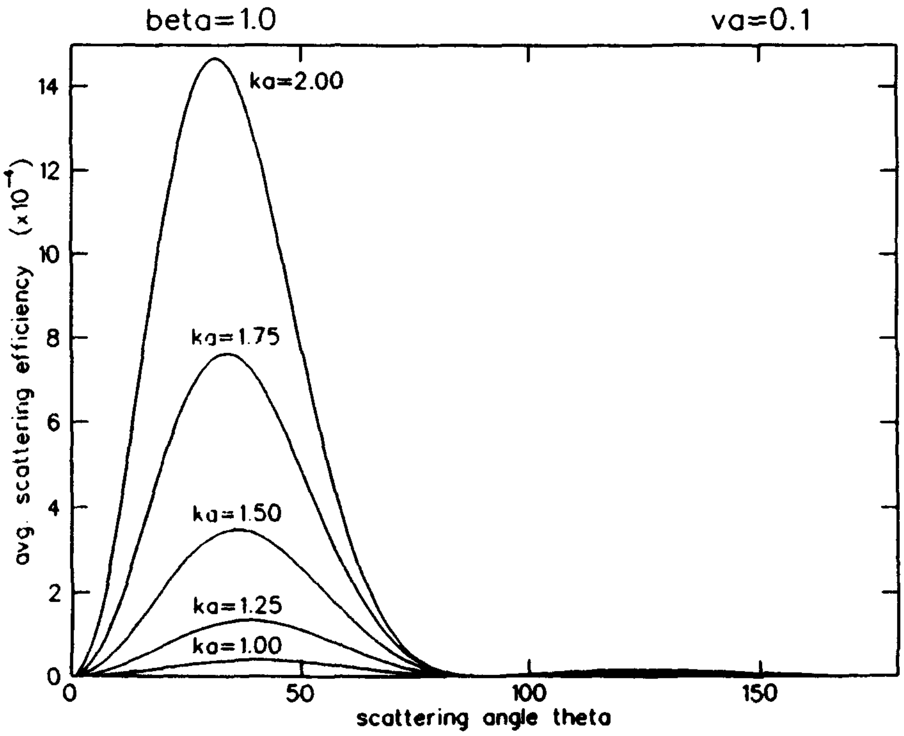

Figure 2. Scattering efficiency vs. scattering angle for a spinning turbule, including only first order contributions for large size parameters.

A plot of $Q^{(1)}{}_s(\vec{k})$ vs. ka is given in Fig. 3, for $v_a = 0.1$.

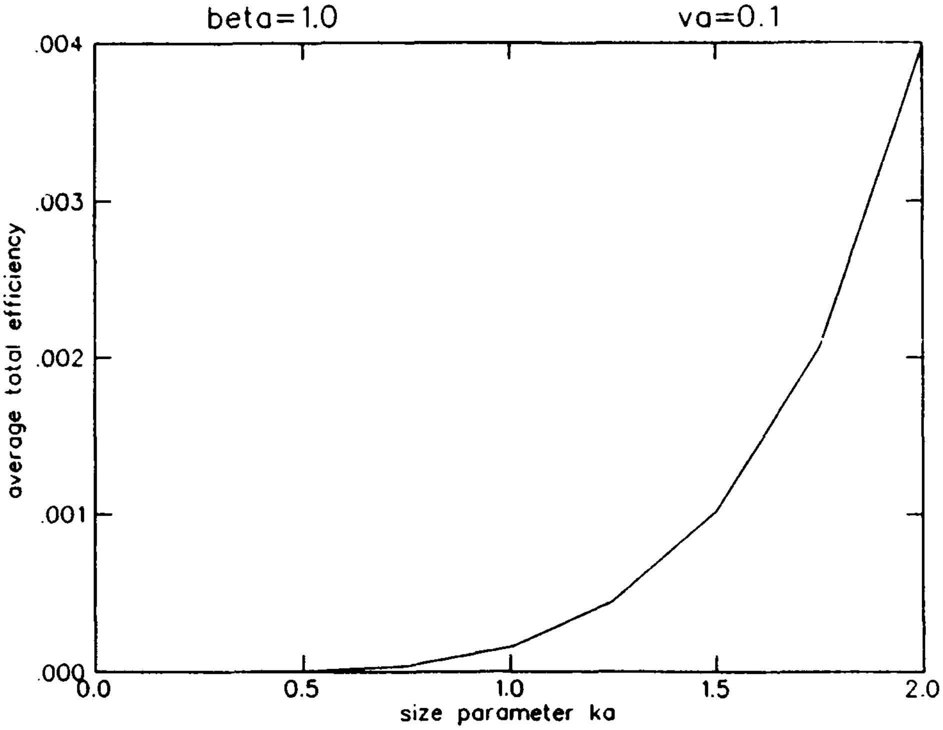

Figure 3. Average scattering efficiency vs. size parameter for a spinning turbule, including only first order contributions.

It is important to look at limits for ka<<1 and ka large. For example, for ka < 0.1, $I_n((ka)^2) \approx 2^{n+1}/(n+1)$. Thus, for ka < 0.1, $Q^{(1)}{}_s(k)$ is proportional to $(ka)^6$. However, for (ka) > 4, $I_n((ka)^2) \approx n!/(ka)^{2n+2}$, so the behavior of $Q^{(1)}{}_s(\vec{k})$ is given by that of the leading term in the sum (65),

$$Q^{(1)}{}_s(\vec{k})\ ka{>}4\ (\pi v^2{}_s/12)(ka)^2. \tag{68}$$

This proportionality to $(ka)^2$ is typical of the Born approximation, and is wrong for very large ka. Just as in electromagnetic scattering, the total scattering efficiency should be independent of ka for very large ka.

3. <u>Second Order Efficiencies</u>. If we combine eqs. (33, 34, 50, 52, 53, 54, 56, and 57-60), we get expressions for the second order scattering efficiencies. These are much too lengthy and complicated to write down in detail. The method is given in Appendix A.

In particular, we calculated only the averages over spin axis orientation. The general form is

$$<Q^{(2)}(\vec{k},\hat{r})> = (v^4_a/512)\ x^2\ e^{-x\zeta} \quad \Sigma^4_{n=0}\ F_n(x)\zeta^n \tag{69}$$

where

$$x \equiv (ka)^2/2\ ,\ \zeta \equiv 1-\cos\theta \tag{70}$$

and the $F_n(x)$ are complicated polynomials in x and x^2. It is these functions that are very lengthy expressions. But, for x << 1, we find simply

$$\begin{aligned} F_o(x) &= (1-\gamma)^2 + (4\beta/3)(1-\gamma) + 8\beta/15 \\ F_1(x) &= 2(1+\beta)(1-\gamma + 2\beta/3) \\ F_2(x) &= (1+\beta)^2 \\ F_3(x) &= F_4(x) = 0, \end{aligned} \tag{71}$$

where we have dropped all terms containing (x, x^2). Here, β is either zero or unity. If we choose β=1, we get the second order result that includes both $f^{(2)}{}_n$ and $f^{(2)}{}_v$, eqs. (53) and (54). If we choose β=0, we get the second order result for a density variation only, where the ψ function is, however, still given by eq. (51).

The expression (69) for $<Q^2(\vec{k},\hat{r})>$, using complete expressions for the $F_n(x)$, is plotted in Figs. 4 through 7 against scattering angle θ, for several values of ka, for v_a = 0.1, for γ = 5/3. Note that its maxima are about two orders of magnitude smaller than those of the first order contributions, as we would expect for small v_a and for these values of ka.

The total scattering efficiency is then given by the integral over solid angle of (69). The result is

$$<Q^{(2)}(k)> = (\pi v^4_a/256)x^2 \quad \Sigma^4_{n=0}\ F_n(x)I_n(x) \tag{72}$$

where the $I_n(x)$ are defined by eq. (67). This scattering efficiency is plotted as a function of ka in Figs. 8 and 9. We note from eqs. (72) and (67) that, for small ka (small x), this is proportional to $(ka)^4$, not $(ka)^6$ as for the first order contribution. This means that, for large enough v_a and small enough ka, the second order contributions could dominate the first order. But for the realistic parameter choices made here, namely v_a = 0.1 and 0.25 < ka < 2, the first order is totally dominant.

Note that there is no interference between first order and second order contributions. Eqs. (52) through (57) reveal that the

first order scattering amplitude is purely imaginary, while the second order ones are purely real. Thus, when we write

$$f = f^{(1)} + f^{(2)} = i|f^{(1)}| + |f^{(2)}|, \tag{73}$$

then we get simply

$$|f|^2 = |f^{(1)}|^2 + |f^{(2)}|^2 \tag{74}$$

Thus, the principal contributions of the flow to the efficiencies go like v^2_a, while the leading contributions of the density variation go like v^4_a.

IV. DISCUSSION

The principal theoretical and calculational results of the work reported are those mentioned in the introduction. We rephrase them somewhat here: i) In order for a quasistatic density variation to exist in an isentropic ideal fluid in the absence of gravitational and Coriolis effects, a quasistatic flow must also exist, and according to eqs. (17) the relative density variation is proportional to the square of the small ratio of flow speed to asymptotic acoustical wavespeed, ii) The fundamental fluid equations yield wave equations (18) for small-amplitude acoustic disturbances superimposed on the quasistatic flow and density variation of a model turbule. These wave equations contain terms linear in the flow speed ratio, which should then produce more scattering than the associated density variations; and iii) The first Born approximation calculations of scattering by a non-uniformly spinning Gaussian model turbule show that the contributions to the scattering of the terms linear in flow speed in the wave equations are indeed about three orders of magnitude larger than the contributions of the terms involving the density variations, for size parameters from 0.25 to 2.0, for flow speed ratios < 0.1. For example, Figs. 3 and 8 show that the first order total scattering efficiency, due to the associated quasistatic density variations alone, is about 10^{-7}, for a size parameter of unit, and a flow speed ratio of 0.1.

Further investigation seems desirable, and is planned. Efforts probably should include i) Realistic modeling of other quasistatic turbules that include flow, such as smoke rings (toroidal vortices). This would require modification of the scattering equations, since such vortices must translate in still air; ii) Extension of the DFG method; iii) Possible application of eikonal approximations similar to those that have been used with great success for large size parameters in quantum mechanics and, recently, in electromagnetic scattering by dielectric particles (Chen, 1989).

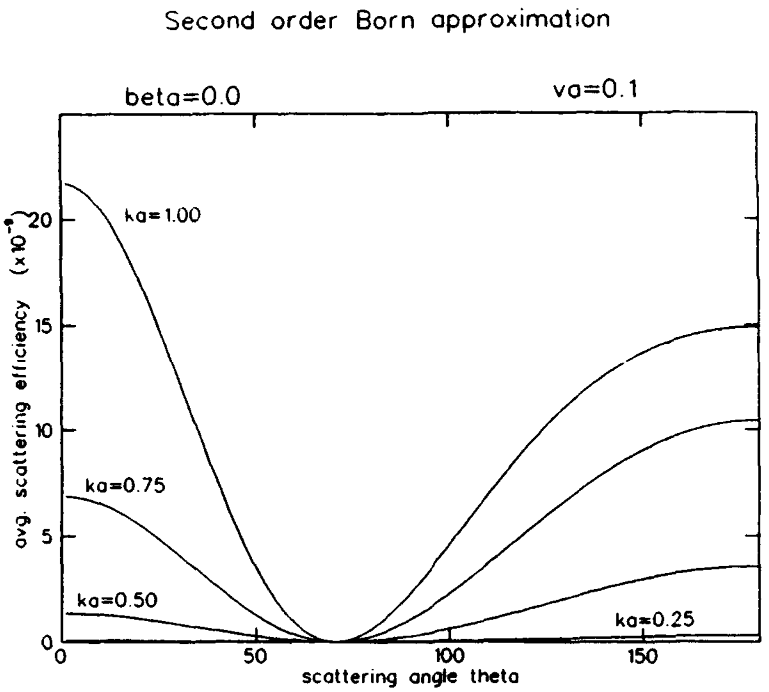

Figure 4. Scattering efficiency vs. scattering angle for small size parameters, for turbule with density variation only (no flow).

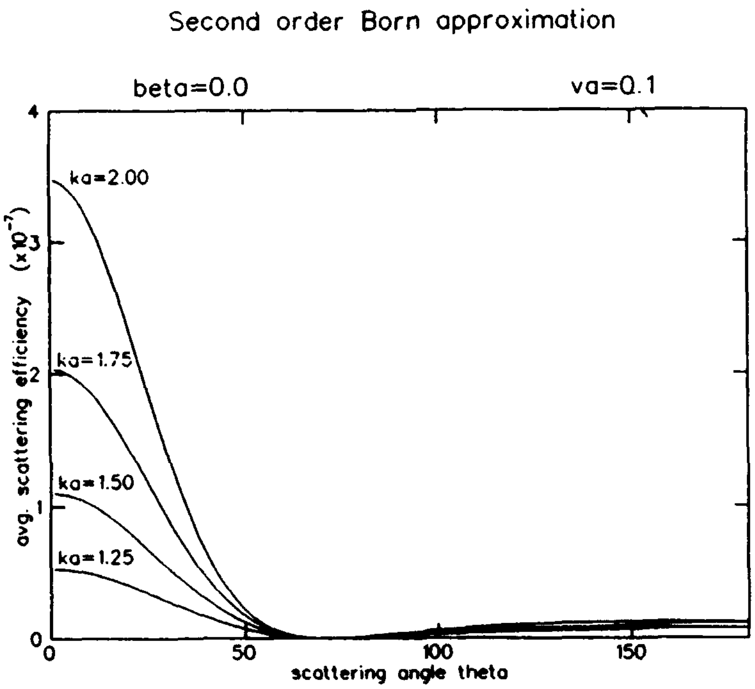

Figure 5. Scattering efficiency vs. scattering angle for large size parameters, for turbule with density variation only (no flow).

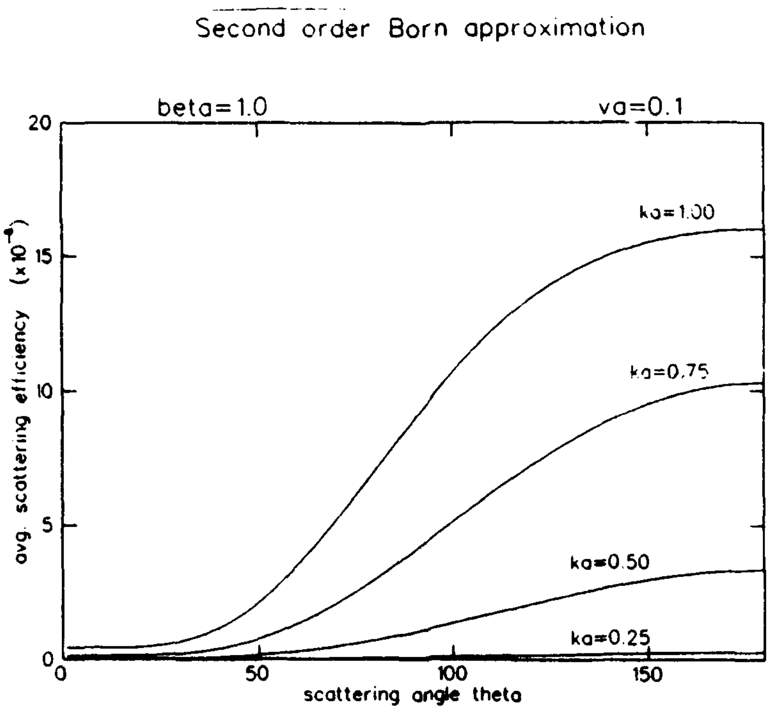

Figure 6. Scattering efficiency vs. scattering angle for size parameters 0.25 to 1.0, for spinning turbule, including only second order contributions.

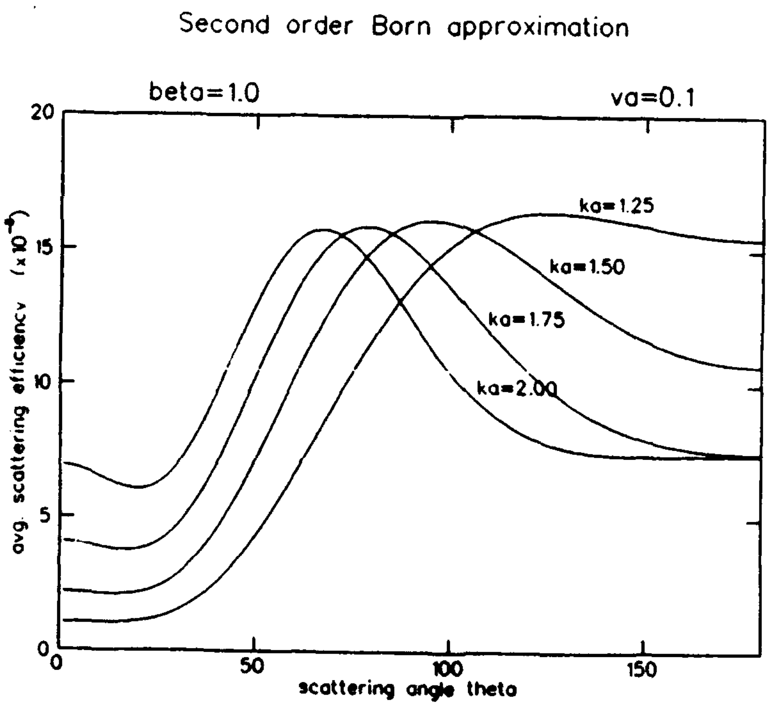

Figure 7. Scattering efficiency vs. scattering angle for size parameters 1.25 to 2.00, for spinning turbule, including only second order contributions.

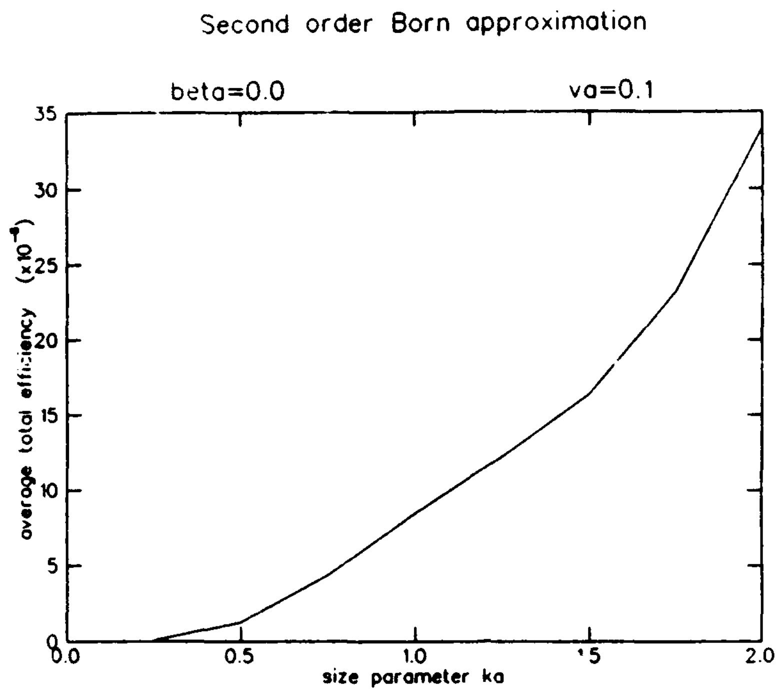

Figure 8. Average scattering efficiency vs. size parameter for the case of density variation only (no flow) in the turbule.

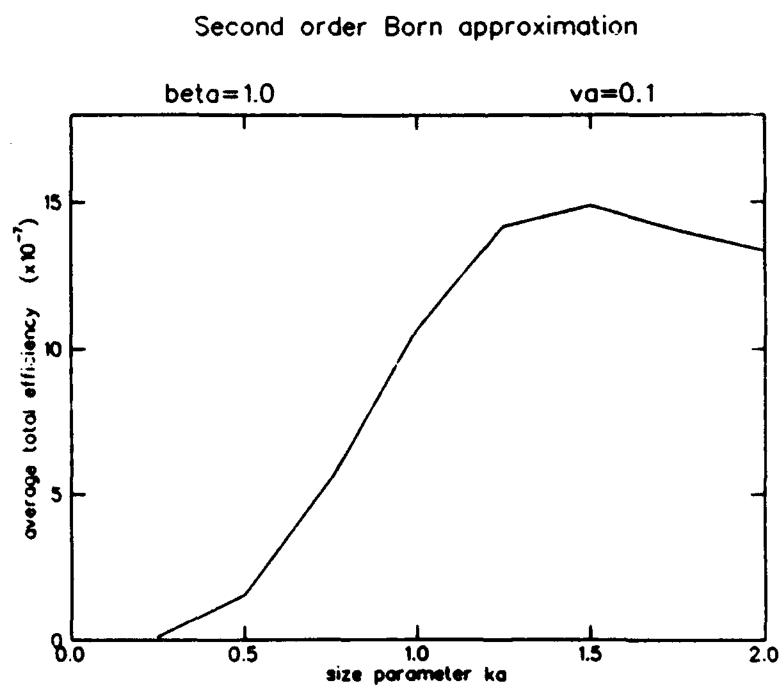

Figure 9. Average scattering efficiency vs. size parameter for a spinning turbule, including only second order contributions.

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APPENDIX A. ORIENTATION-AVERAGED CROSS SECTIONS

The total second order scattering amplitude of eqs. (53) and (54),

$f^{(2)} = f_n^{(2)} + f_v^{(2)}$, may be written as

$$f^{(2)} = (\alpha^2/8\pi k)\ J(x,\xi)\ F(x,\xi,\varphi,\theta_\alpha) \tag{A1}$$

where

$$\xi \equiv 1 - \cos\theta,\ x = (ka)^2/2 \tag{A2}$$

$$J(x,\xi) \equiv \pi^{3/2}\ x^{5/2}\ e^{-x\xi} \tag{A3}$$

$$F(x,\xi,\varphi,\theta_\alpha) = C(x,\xi) + S(x,\xi,\varphi,\theta_\alpha) \tag{A4}$$

$$C(x,\xi) = A(\xi)(2-x\xi) \tag{A5}$$

$$A(\xi) = \tfrac{1}{2}[1-\gamma+\xi(1+\beta)] \tag{A6}$$

$$S(x,\xi,\varphi,\theta_\alpha) = \tfrac{1}{2}\ x\ A(\xi)D(\xi,\varphi,\theta_\alpha) + \beta\sin^2\theta_\alpha(1-\tfrac{1}{2}x\sin^2\theta\sin^2\varphi) \tag{A7}$$

$$D(\xi,\varphi,\theta_\alpha) = \xi^2\cos^2\theta_\alpha + \sin^2\theta\sin^2\theta_\alpha\cos^2\varphi - 2\xi\sin\theta\sin\theta_\alpha\cos\theta_\alpha\cos\varphi \tag{A8}$$

We want to calculate $\langle F^2\rangle$, where

$$\langle G(\theta_\alpha,\varphi)\rangle \equiv (4\pi)^{-1} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta_\alpha\ \sin\theta_\alpha\ G(\theta_\alpha,\varphi) \tag{A9}$$

with $G(\theta_\alpha,\varphi)$ any functions of these variables.

We see that

$$\langle F^2\rangle = C^2 + 2C\langle S\rangle + \langle S^2\rangle \tag{A10}$$

In order to evaluate the various averages, we need

$$\begin{aligned} &\langle\cos\varphi\rangle = \langle\sin\varphi\rangle = 0, && \langle\cos^2\varphi\rangle = \langle\sin^2\varphi\rangle = \tfrac{1}{2} \\ &\langle\sin^2\theta_\alpha\rangle = 2/3, && \langle\cos^2\theta_\alpha\rangle = 1/3 \\ &\langle\sin^4\theta_\alpha\rangle = 8/15, && \langle\cos^4\theta_\alpha\rangle = 3/15 \\ &\langle\sin^2\theta_\alpha\cos^2\theta_\alpha\rangle = 2/15, && \langle\cos^4\varphi\rangle = \langle\sin^4\varphi\rangle = 3/8 \end{aligned} \tag{A11}$$

From these, we get

$$\langle D\rangle = 2\xi/3 \ , \quad \langle D^2\rangle = 4\xi^2/5$$

$$\langle D\sin^2\theta_\alpha\rangle = 8\xi/15 - 2\xi^2/15 \tag{A12}$$

$$\langle D\sin^2\theta_\alpha \sin^2\varphi\rangle = 2\xi/15$$

Then, we get

$$\langle S\rangle = (2\beta/3)[1-x(2\xi-\xi^2)/4] \tag{A13}$$

$$\langle S^2\rangle = x^2A^2\xi^2/5 + 8\beta/15 - (4\beta/15)(2\xi-\xi^2)x \tag{A14}$$

$$+ \beta(2\xi-\xi^2)^2\, x^2/20 + \beta Ax(8\xi/15 - 2\xi^2/15-\xi(2\xi-\xi^2)x/15)$$

Then

$$2C\langle S\rangle = \beta(2-x\xi)(4A/3)[1-x(2\xi-\xi^2)/4] \tag{A15}$$

$$C^2 = A^2(2-x\xi)^2 \tag{A16}$$

If we combine these according to eq. (A10), we may rearrange terms so that the result looks like

$$\langle F^2\rangle = \Sigma^4_{n=0}\ F_n(x)\,\xi^n \tag{A17}$$

Using (A2,A1) and $\alpha = (v_a)/(ka)$, $x = (ka)^2/2$, yields the expression (69) of the text.

APPENDIX B. SUMMARY OF RELEVANT FORMULAS

A model spinning turbule was chosen to have the following quasistatic flow velocity $\vec{v}(\vec{r})$ and molecular number density $n_0(\vec{r})$:

$$\vec{v}(\vec{r}) = (\vec{\Omega}\times\vec{r})\exp(-r^2/a^2) \tag{B1}$$

$$n_0(\vec{r}) = -n_\infty v^2/(2c_\infty^2) \tag{B2}$$

This represents a non-uniformly spinning turbule, with angular velocity parameter Ω about an axis $\hat{\Omega}$ through the origin of coordinates. The quantities (n_∞, c_∞) are the background (number density, wavespeed) far from the turbule; the parameter a is an "effective radius" of the turbule; $\vec{r}$ is the position vector. The expression for $n_0(\vec{r})$ follows approximately from the assumption that turbules are formed adiabatically.

If an acoustic plane wave with propagation vector $\vec{k}$, where $k = 2\pi/\lambda = \omega/c_\infty$, ω = angular frequency, is incident upon this model turbule, then the scattering amplitude $f(\vec{k}, \hat{r})$ in first Born approximation is given by

$$f(\vec{k},\hat{r}) = f^{(1)} + f_n^{(2)} + f_v^{(2)} \tag{B3}$$

where $f^{(1)}$ is linear in Ω, and $f^{(2)}$ is quadratic in Ω. The differential scattering cross section then turns out to be given by

$$\sigma(\vec{k},\hat{r}) = |f^{(1)}|^2 + |f_n^{(2)} + f_v^{(2)}|^2 \equiv \sigma^{(1)} + \sigma^{(2)} \tag{B4}$$

The expressions for $f^{(1)}$ and $f^{(2)}$ are as follows:

$$f^{(1)}(\vec{k},\hat{r}) = -(\alpha^2/2\pi k)\cos\theta\, e_{jki}\,\Omega_k J_i \tag{B5}$$

$$f_n^{(2)}(\vec{k},\hat{r}) = (\alpha^2/4\pi k)(1-\gamma+K^2/2)^{1/2}(\delta_{ij}-\Omega_i\Omega_j)J_{ij} \tag{B6}$$

$$f_v^{(2)}(\vec{k},\hat{r}) = (\alpha^2/4\pi k)\{(K^2/4)(\delta_{ij} - \Omega_i\Omega_j) + \hat{k}_r\hat{k}_s\Omega_l\Omega_p\epsilon_{rli}\epsilon_{spj}\}J_{ij} \tag{B7}$$

where

$$\alpha = \Omega/\omega \tag{B8}$$

$$J_i = i\,\tfrac{1}{2}\,\pi^{3/2}\,(ka)^5\,K_i\,\exp(-(ka)^2K^2/4) \tag{B9}$$

$$J_{ij} = \tfrac{1}{4}(ka)^5\,[\delta_{ij} - \tfrac{1}{4}(ka)^2K_iK_j](\pi/2)^{3/2}\,\exp(-(ka)^2K^2/8 \tag{B10}$$

and

$$K_i \equiv \hat{k}_i - \hat{r}_i,\ K^2 = 2(1 - \cos\theta), \tag{B11}$$

when θ is the polar scattering angle, the angle between $\hat{k}$ and the observation direction $\hat{r} = \hat{e}_1\sin\theta\cos\phi + \hat{e}_2\sin\theta\sin\phi + \hat{e}_3\cos\theta$. These equations (B5) through (B11) are the same as eqs. (50) and (52) through (57) of the text. Here ϕ is the azimuthal scattering angle.

The differential cross section is given by inserting (B5) through (B7) into (B4). It depends on the azimuthal angle ϕ as well as on the polar angle θ. For example, the scattering efficiency from $\sigma^{(1)} = |f^{(1)}|^2$ is

$$\begin{aligned} Q^{(1)}(k,\hat{r}) &\equiv \sigma^{(1)}/\pi a^2 \\ &= (\Omega a/4c_\infty)^2(ka)^6[\sin\theta\cos\theta\sin\theta_\Omega\sin(\phi - \phi_\Omega)]^2 \end{aligned} \tag{B12}$$

where $(\theta_\Omega,\phi_\Omega)$ are the (polar, azimuthal) angles of the spin vector $\hat{\Omega}$.

There is interest in the average of these differential cross sections over random orientations of the spin axis. These were obtained in two ways, analytic and numerical. The analytic method is outlined in the text and in Appendix A. The numerical method simply numerically averages over uniformly spaced values of $\cos\theta_\Omega$ and ϕ_Ω. The results of this latter method agreed with those of the former, which are displayed as Figs. 1 through 9 of the text. The analytic relation is

$$\langle Q^{(1)}(k,\hat{r})\rangle = (2/3)(\Omega a/4c_\infty)^2(ka)^6(\sin\theta\cos\theta)^2 \tag{B13}$$

If E_i is the incident exitance (watt.m^{-2}) of the sound field and $I(\theta)$ is the intensity (watt.sr^{-1}) of the remote scattered field at the off-axis angle θ, then

$$I(\theta) = \pi a^2 \langle Q^{(1)}(k,\hat{r})\rangle E_i \qquad \text{(B14)}$$

The cross section $\sigma^{(2)}$ is much smaller than $\sigma^{(1)}$, for realistic choices of the parameter $\Omega a/c_\infty \ll 1$.

The total cross sections, defined by

$$\sigma^{(i)}(k,\hat{r}) = \int d\Omega\, \sigma^{(i)}(k,\hat{r}),\ i = 1,2 \qquad \text{(B15)}$$

or rather the scattering efficiencies $Q^{(i)} = \sigma^{(i)}/\pi a^2$, were also obtained both analytically, as described in the text and in appendix A, and numerically, by numerical integration over (θ,ϕ). Here, $d\,\Omega = \sin\theta\, d\,\theta d\,\phi$ = solid angle element. The results agreed. The analytic total scattering efficiency for $\sigma^{(1)}$ is

$$\langle Q^{(1)}(k)\rangle = (\pi/45)\,(\Omega a/c_\infty)^2\,(ka)^6 \qquad \text{(B16)}$$

Some authors, especially in radar scattering, relate the differential scattering cross section to the ideal isotropic scatterer. The scattered field from the ideal isotropic scatterer is the same for all scattering angles θ with magnitude of the total cross section times the incident exitance divided by 4π. The actual field from a non-isotropic scatterer is then the magnitude of the previous sentence times an appropriate function of θ called the phase function. This information is provided to alert the reader that there is a quantity of 4π applied in the radar scattering concept that is not needed in the development of this report.

TEXT MODE REPRESENTATIONS

Ω

$\underline{\Omega}$

Ω

$\hat{\Omega}$

$\hat{\Omega}$

$\wedge$

$\wedge$

GRAPHICS MODE REPRESENTATIONS

$\boldsymbol{\Omega}$ OMEGA bold

$\underline{\Omega}$ OMEGA underline

$\hat{\Omega}$ OMEGA hat

$\vec{\Omega}$ OMEGA vec

$\bar{\Omega}$ OMEGA bar

$\overset{\hat{}}{\Omega}$ stack {^#Ω}

$\overset{\wedge}{\Omega}$ stack {Λ#Ω}

$\overset{\wedge}{\Omega}$ stack {Λ#Ω}

$\overset{\wedge}{\Omega}$ stack {Λ#Ω}

$\overset{\wedge}{\Omega}$ func stack {Λ#Ω}

$\overset{\wedge}{\Omega} \quad \overset{\wedge}{\Omega}$

*rtuvWERr*tuvWER stack {func Λ# func Ω}